

CORRIGENDUM TO “ON INJECTIVE MODULES AND SUPPORT VARIETIES FOR THE SMALL QUANTUM GROUP”

CHRISTOPHER M. DRUPIESKI

ABSTRACT. The proof of Theorem 5.12 in [1] does not make sense as written because the algebra $u_\zeta(\mathfrak{b}_\alpha^+)$ need not be a Hopf subalgebra of $u_\zeta(\mathfrak{b}^+)$ unless α is a simple root. This note describes how the proof should be modified to work around this fact.

All notation is taken from [1]. Theorem 5.12 of [1] is as follows:

Theorem 5.12. *Let M be a finite-dimensional $u_\zeta(\mathfrak{b}^+)$ -module and let $\alpha \in \Phi^+$. Then the root vector $e_\alpha \in \mathfrak{u}^+$ is an element of $\mathcal{V}_{u_\zeta(\mathfrak{b}^+)}(M)$ if and only if M is not projective for $u_\zeta(e_\alpha)$.*

The proof given in [1] involves considering the restriction of M to a certain subalgebra $u_\zeta(\mathfrak{b}_\alpha^+)$ of $u_\zeta(\mathfrak{b}^+)$. However, the proof does not make sense as written because, while $u_\zeta(\mathfrak{b}_\alpha^+)$ does admit the structure of a Hopf algebra, it need not be a Hopf subalgebra of $u_\zeta(\mathfrak{b}^+)$ unless α is a simple root. The purpose of this note is to give a correct proof of the theorem.

Proof of Theorem 5.12. Let $\alpha \in \Phi^+$ be a positive root. Write $\alpha = \sum_{\beta \in \Pi} m_\beta \beta$ as a sum of simple roots, and set $K_\alpha = \prod_{\beta \in \Pi} K_\beta^{m_\beta}$. Now let $u_\zeta(\mathfrak{b}_\alpha^+)$ be the subalgebra of $u_\zeta(\mathfrak{b}^+)$ generated by K_α and E_α . If α is a simple root, then K_α and E_α are just the defining generators of $u_\zeta(\mathfrak{g})$ labeled by α . More generally, let $\beta \in \Pi$ be a simple root of the same length as α . Then the assignments $E_\beta \mapsto E_\alpha$ and $K_\beta \mapsto K_\alpha$ extend to an isomorphism of algebras $u_\zeta(\mathfrak{b}_\beta^+) \cong u_\zeta(\mathfrak{b}_\alpha^+)$. The algebra $u_\zeta(\mathfrak{b}_\beta^+)$ is a Hopf subalgebra of $u_\zeta(\mathfrak{b}^+)$, so this shows that $u_\zeta(\mathfrak{b}_\alpha^+)$ admits the structure of a Hopf algebra (though it may not be a Hopf subalgebra of $u_\zeta(\mathfrak{b}^+)$ unless α is a simple root). In particular, taking $H = u_\zeta(\mathfrak{b}_\alpha^+)$ and $D = u_\zeta(e_\alpha)$, it follows from [1, Lemma 3.3] that a finite-dimensional $u_\zeta(\mathfrak{b}_\alpha^+)$ -module M is injective (equivalently, projective) if and only if its restriction to $u_\zeta(e_\alpha)$ is injective (respectively, projective).

Now let M be a finite-dimensional $u_\zeta(\mathfrak{b}^+)$ -module. Since $u_\zeta(\mathfrak{b}^+)$ is a Hopf algebra, the dual space M^* is naturally a $u_\zeta(\mathfrak{b}^+)$ -module, and then so is the tensor product $V := M \otimes M^*$. More explicitly, the natural isomorphism $M \otimes M^* \cong \text{Hom}_k(M, M)$ is an isomorphism of $u_\zeta(\mathfrak{b}^+)$ -modules when the $u_\zeta(\mathfrak{b}^+)$ -module structure on $\text{Hom}_k(M, M)$ is defined via the formula in [1, Remark 2.2]. We define the $u_\zeta(\mathfrak{b}_\alpha^+)$ -action on V to be the restriction of the $u_\zeta(\mathfrak{b}^+)$ -action.

Next, recall that each irreducible $u_\zeta(\mathfrak{b}^+)$ -module is one-dimensional of some weight $\lambda \in X$. More precisely, the irreducible $u_\zeta(\mathfrak{b}^+)$ -modules are indexed by the elements of the quotient group $X/\ell X$. Denote the irreducible $u_\zeta(\mathfrak{b}^+)$ -module of weight λ by k_λ . Then considering a composition series for the $u_\zeta(\mathfrak{b}^+)$ -module M^* , it follows that V admits a $u_\zeta(\mathfrak{b}^+)$ -module filtration with sections of the form $M \otimes k_\lambda$. The subalgebra $u_\zeta(\mathfrak{u}^+)$ of $u_\zeta(\mathfrak{b}^+)$ acts trivially on k_λ , so it follows from [1, Corollary 5.5] that V admits a $u_\zeta(e_\alpha)$ -module filtration with each section isomorphic to M .

Now suppose that M is projective (equivalently, injective) as a $u_\zeta(e_\alpha)$ -module. Since V admits a $u_\zeta(e_\alpha)$ -module filtration with sections isomorphic to M , we conclude that V is injective for $u_\zeta(e_\alpha)$. Then by [1, Corollary 5.11], there exists $r \in \mathbb{N}$ such that $x_\alpha^r \in J_{u_\zeta(\mathfrak{b}^+)}(M)$, so $e_\alpha \notin \mathcal{V}_{u_\zeta(\mathfrak{b}^+)}(M)$.

In preparation for the converse, recall that the structure of the cohomology ring $H^\bullet(u_\zeta(\mathfrak{b}_\alpha^+), k)$ and the right $H^\bullet(u_\zeta(\mathfrak{b}_\alpha^+), k)$ -module structure of the cohomology group $H^\bullet(u_\zeta(\mathfrak{b}_\alpha^+), V)$ depend only

2000 *Mathematics Subject Classification.* Primary 17B37; Secondary 20G10.

The author is thankful to Alexey Sevastyanov for pointing out the flaw in the proof of [1, Theorem 5.12].

on the algebra structure of $u_\zeta(\mathfrak{b}_\alpha^+)$ and on the structure of V as a left $u_\zeta(\mathfrak{b}_\alpha^+)$ -module. In particular, they are independent of the existence of a Hopf algebra structure on $u_\zeta(\mathfrak{b}_\alpha^+)$. Using the fact that $u_\zeta(\mathfrak{b}_\alpha^+)$ admits the Hopf algebra structure of $u_\zeta(\mathfrak{b}_\beta^+)$, we can deduce from the calculation of $H^\bullet(u_\zeta(\mathfrak{b}^+), k)$ when \mathfrak{b}^+ is a Borel subalgebra of \mathfrak{sl}_2 that $H^\bullet(u_\zeta(\mathfrak{b}_\alpha^+), k)$ is a polynomial algebra generated in cohomological degree 2; cf. [1, Theorem 5.2]. Let us write $H^\bullet(u_\zeta(\mathfrak{b}_\alpha^+), k) \cong k[e_\alpha^*]$, considering $k[e_\alpha^*]$ as the algebra of polynomial functions on the subspace of \mathfrak{u}^+ spanned by e_α . One can show that the restriction map $H^\bullet(u_\zeta(\mathfrak{b}^+), k) \rightarrow H^\bullet(u_\zeta(\mathfrak{b}_\alpha^+), k)$ induced by the inclusion $u_\zeta(\mathfrak{b}_\alpha^+) \hookrightarrow u_\zeta(\mathfrak{b}^+)$ then identifies with the natural restriction map $S(\mathfrak{u}^{+*}) \rightarrow k[e_\alpha^*]$, i.e., with the map that restricts functions from \mathfrak{u}^+ to the space ke_α .¹

Now suppose that $e_\alpha \notin \mathcal{V}_{u_\zeta(\mathfrak{b}^+)}(M)$. By [2, Proposition 2.4(4)], the support variety $\mathcal{V}_{u_\zeta(\mathfrak{b}^+)}(M)$ is a union of relative support varieties:

$$\mathcal{V}_{u_\zeta(\mathfrak{b}^+)}(M) = \bigcup_{\lambda \in X/\ell X} \mathcal{V}_{u_\zeta(\mathfrak{b}^+)}(k_\lambda, M).$$

For each $u_\zeta(\mathfrak{b}^+)$ -module N one has $\mathcal{V}_{u_\zeta(\mathfrak{b}^+)}(N, M) = \mathcal{V}_{u_\zeta(\mathfrak{b}^+)}(k, M \otimes N^*)$ because of a corresponding isomorphism at the level of extension groups, so $e_\alpha \notin \mathcal{V}_{u_\zeta(\mathfrak{b}^+)}(k, M \otimes k_\lambda^*)$ for each $\lambda \in X$. For the rest of the proof, redefine V to be the $u_\zeta(\mathfrak{b}^+)$ -module $M \otimes k_\lambda^*$.

Let $J_\alpha(k, M \otimes k_\lambda^*)$ be radical of the annihilator ideal for the right action of the cohomology ring $H^\bullet(u_\zeta(\mathfrak{b}_\alpha^+), k) \cong k[e_\alpha^*]$ on $H^\bullet(u_\zeta(\mathfrak{b}_\alpha^+), M \otimes k_\lambda^*)$. Let $\mathcal{V}_\alpha(k, M \otimes k_\lambda^*)$ be the (conical) subvariety of ke_α defined by $J_\alpha(k, M \otimes k_\lambda^*)$. Since the restriction map $H^\bullet(u_\zeta(\mathfrak{b}^+), k) \rightarrow H^\bullet(u_\zeta(\mathfrak{b}_\alpha^+), k)$ is surjective, it induces by naturality a closed embedding $\mathcal{V}_\alpha(k, M \otimes k_\lambda^*) \hookrightarrow \mathcal{V}_{u_\zeta(\mathfrak{b}^+)}(k, M \otimes k_\lambda^*)$. Moreover, by the assertion at the end of the previous paragraph we can identify the image of $\mathcal{V}_\alpha(k, M \otimes k_\lambda^*)$ with a conical subset of $\mathcal{V}_{u_\zeta(\mathfrak{b}^+)}(k, M \otimes k_\lambda^*) \cap ke_\alpha$. Since $e_\alpha \notin \mathcal{V}_{u_\zeta(\mathfrak{b}^+)}(k, M \otimes k_\lambda^*)$, this conical subset must be $\{0\}$. Then we must have $\mathcal{V}_\alpha(k, M \otimes k_\lambda^*) = \{0\}$ as well.

Up to this point we have considered $M \otimes k_\lambda^*$ as a $u_\zeta(\mathfrak{b}_\alpha^+)$ -module by way of restriction from $u_\zeta(\mathfrak{b}^+)$, with $u_\zeta(\mathfrak{b}^+)$ acting diagonally on $M \otimes k_\lambda^*$. On the other hand, since $u_\zeta(\mathfrak{b}_\alpha^+)$ admits the Hopf algebra structure of $u_\zeta(\mathfrak{b}_\beta^+)$, we could use the $u_\zeta(\mathfrak{b}_\alpha^+)$ -module structure on M , obtained via restriction from $u_\zeta(\mathfrak{b}^+)$, and the $u_\zeta(\mathfrak{b}_\alpha^+)$ -module structure on k_λ^* , coming from the Hopf algebra structure on $u_\zeta(\mathfrak{b}_\alpha^+)$, together with the Hopf algebra structure on $u_\zeta(\mathfrak{b}_\alpha^+)$ to define the diagonal action of $u_\zeta(\mathfrak{b}_\alpha^+)$ on $M \otimes k_\lambda^* = M \otimes k_{-\lambda}$. However, in both situations one has $M \otimes k_{-\lambda} \cong M$ as a $u_\zeta(e_\alpha)$ -module, from which it follows that the two $u_\zeta(\mathfrak{b}_\alpha^+)$ -actions on $M \otimes k_\lambda^*$ are the same. So now considering $u_\zeta(\mathfrak{b}_\alpha^+)$ as a Hopf algebra, we deduce for each $\lambda \in X$ that

$$\{0\} = \mathcal{V}_\alpha(k, M \otimes k_\lambda^*) = \mathcal{V}_{u_\zeta(\mathfrak{b}_\alpha^+)}(k, M \otimes k_\lambda^*) = \mathcal{V}_{u_\zeta(\mathfrak{b}_\alpha^+)}(k_\lambda, M),$$

and hence by [2, Proposition 2.4(4)] that $\mathcal{V}_{u_\zeta(\mathfrak{b}_\alpha^+)}(M) = \{0\}$. Then by [2, Proposition 2.4(1)], M is projective as a $u_\zeta(\mathfrak{b}_\alpha^+)$ -module, which implies that M is projective for $u_\zeta(e_\alpha)$. \square

REFERENCES

1. C. M. Drupieski, *On injective modules and support varieties for the small quantum group*, Int. Math. Res. Not. (2011), no. 10, 2263–2294.
2. J. Feldvoss and S. Witherspoon, *Support varieties and representation type of small quantum groups*, Int. Math. Res. Not. (2010), no. 7, 1346–1362.
3. V. Ginzburg and S. Kumar, *Cohomology of quantum groups at roots of unity*, Duke Math. J. **69** (1993), no. 1, 179–198.

DEPARTMENT OF MATHEMATICAL SCIENCES, DEPAUL UNIVERSITY, CHICAGO, IL 60614, USA
E-mail address: cdrupies@depaul.edu

¹This description of the restriction homomorphism can be verified by calculating both cohomology rings via the argument in [3, §2] and verifying at each step in the argument that the calculations are compatible with restriction from $u_\zeta(\mathfrak{b}^+)$ to $u_\zeta(\mathfrak{b}_\alpha^+)$.